Tentamen Numerical Mathematics 2
January 25, 2017

Duration: 3 hours.
In front of the questions one finds the weights used to determine the final mark.

Problem 1

a. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$$

for the problem $Ax = b$, with $\epsilon = 10^{-10}$. For this problem we want to find the least-squares solution.

(i) [3] Show that it is not possible to solve this problem via the normal equations on a standard PC.

(ii) [5] Create a QR factorization of A using Housholder transformations. How is this QR factorization used to find the least squares solution and does this lead to a solvable system?

(iii) [4] Compute the singular values of A.

b. Suppose we make an LU factorization of a matrix A of order n where the matrix L has ones on its diagonal and moreover L is diagonal dominant by rows.

(i) [3] Let e_1 be the vector $[1, 0, 0, \cdots, 0]^T$. Show that all the entries the solution y of $Ly = e_1$, are less than or equal to 1 in magnitude. (Hint, prove by induction.)

(ii) [3] Show that every entry of the inverse of L is less than or equal to one.

(iii) [1] Show that $||U||_\infty \leq n||A||_\infty$

(iv) [1] Solving $Ax = b$ with the above LU factorization one can show that we in fact find a solution $\hat{x}$ which is the exact solution of a system $(A + \delta A)\hat{x} = b$ where $|\delta A| \leq nu(3|A| + 5|L||U|) + O(u^2)$. Show that $||\delta A||_\infty \leq nu(3||A||_\infty + 10||U||_\infty) + O(u^2)$.

(v) [1] Bound $||\delta A||_\infty$ in $||A||_\infty$.

(vi) [2] How is this expression used to bound the relative error in x?

Problem 2

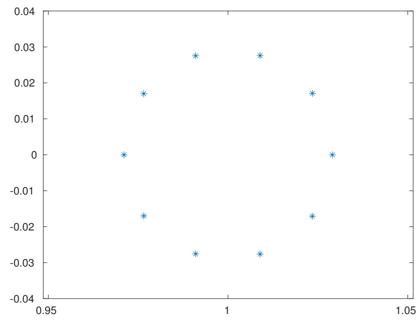
a. Consider the $n \times n$, n even, matrix

$$A(\epsilon) = \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ \epsilon & & & 1 \end{bmatrix}$$

(i) [3] Show that the characteristic equation of $A(\epsilon)$ is given by $(\lambda - 1)^n - \epsilon = 0$.

(ii) [4] Consider the problem $(\lambda - 1)^n = d$. Give the absolute condition number of this problem for $d \neq 0$. What happens to this condition number if d tends to zero?

(iii) [3] In the figure right the computed eigenvalues of $Q^T A(0) Q$, where Q is a random orthogonal matrix, are depicted for $n = 10$. Relate the results to the solution of the characteristic equation of part (i) where ϵ is the unit round.



b. The QR-method to find all eigenvalues of a matrix A is defined by the following iteration

$$
\begin{aligned}
A_0 &= A \\
Q_i R_i &= A_i, \text{ for } i = 0, 1, 2, 3, \ldots \\
A_{i+1} &= R_i Q_i
\end{aligned}
$$

(i) [2] Show that A_i is similar to A for all i.

(ii) [4] Let $\hat{Q}_{i-1} \hat{R}_{i-1} = A^i$ (indeed A to the power i). Show that $A_i = (\hat{Q}_{i-1})^T A \hat{Q}_{i-1}$. (Hint, prove by induction).

(iii) [4] Assume furthermore that A is symmetric and tridiagonal. Show that A_i is tridiagonal for all i.

c. [4] Show the existence of the generalized Schur form

$$
\begin{aligned}
AZ &= YS \\
BZ &= YT
\end{aligned}
$$

where both Z and Y are orthogonal and S, T upper triangular. You may assume that A and B are non-singular. You may start from the standard Schur form of $B^{-1}A$. How is the generalized Schur form used to solve the generalized eigenvalue problem $Ax = \lambda Bx$?

Problem 3

Consider the basis $\{1, x, x^2, x^3, \cdots,\}$ on the interval $[0, 1]$. Moreover on this interval an inner product is defined by $(f, g) = \int_0^1 f(x) g(x) dx$.

a. [4] Derive the first three orthogonal basis functions (so up to the quadratic function).

b. [2] Show that the zeros of the quadratic basis function are $\frac{1}{2} \pm \frac{1}{6}\sqrt{3}$.

c. [4] Show that for the original (so the one on top of this question) basis, using n terms, the least squares approximates of a function f on the subspace is found by solving the coefficients from the linear system $Ac = b$ with $a_{ij} = 1/(i+j+1)$ and $b_i = (f, x^i)$.

d. [3] Suppose we now use the orthogonal basis $\{\psi_0(x), \psi_1(x), \cdots, \psi_n(x)\}$. To which linear system does the least squares minimization lead here?

e. [3] The minimizations in both part b and c lead to the same polynomial approximation. Why is this the case?

f. [2] Numerically, using the orthogonal polynomials as a basis is favored. Why?

g. [3] A Gauss method for the integration of

$$\frac{dy}{dt} = f(t, y)$$

is given by the Butcher tableau

$$\begin{array}{c|cc} \frac{1}{2} - \frac{1}{6}\sqrt{3} & \frac{1}{4} & \frac{1}{4} - \frac{1}{6}\sqrt{3} \\ \frac{1}{2} + \frac{1}{6}\sqrt{3} & \frac{1}{4} + \frac{1}{6}\sqrt{3} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Explain why in the first column one finds the zeros of the polynomial as given in part b. What will be the order of accuracy of this method.

h. [2] Let $f(t, y) = y(y + 1)$. Which system has to be solved in each time step using the Gauss method from the previous part?

Qu 1

(i) normal equations

$$Ax = b \rightarrow A^T A x = A^T b \quad (1)$$

$\varepsilon^2 < 10^{-16}$

1 3

$$A^T A = \begin{bmatrix} 1 & \varepsilon & 0 \\ 1 & 0 & \varepsilon \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} 1+\varepsilon^2 & 1 \\ 1 & 1+\varepsilon^2 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(2) is singular

(ii) $\hat{H}_1 \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} \sqrt{1+\varepsilon^2} \\ 0 \end{bmatrix}$

1 $\hat{H}_1 = (1 - 2ww^T)$

$$\hat{w} = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} \pm \begin{bmatrix} 1+\varepsilon^2 \\ 0 \end{bmatrix}$$

Choose +

$$\hat{w} = \begin{bmatrix} 2+\varepsilon^2 \\ \varepsilon \end{bmatrix} \cong \begin{bmatrix} 2 \\ \varepsilon \end{bmatrix} \quad \text{ignore } \varepsilon^2 \text{ terms}$$

$$w = \begin{bmatrix} 1 \\ \varepsilon/2 \end{bmatrix}$$

~~$H_1 = I$~~

5

$$\hat{H}_1 \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} - 2\begin{bmatrix} 1 \\ \varepsilon/2 \end{bmatrix}(1+\varepsilon^2) = \begin{bmatrix} -1-2\varepsilon^2 \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\hat{H}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ \varepsilon/2 \end{bmatrix} \cdot 1 = \begin{bmatrix} -1 \\ -\varepsilon \end{bmatrix}$$

$$\begin{bmatrix} \hat{H}_1 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -\varepsilon \\ 0 & \varepsilon \end{bmatrix}$$

$$\varepsilon \hat{H}_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \pm \begin{bmatrix} \varepsilon\sqrt{2} \\ 0 \end{bmatrix} \qquad w = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \pm \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

Kies − $\quad w = \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & \hat{H}_2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -\varepsilon \\ 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & \varepsilon\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$\underbrace{H_2 H_1}_{Q^T} A = \underbrace{\begin{bmatrix} -1 & -1 \\ 0 & \varepsilon\sqrt{2} \\ 0 & 0 \end{bmatrix}}_{R}, \qquad Ax = b \quad 2$$

$$\begin{bmatrix} -1 & -1 \\ 0 & \varepsilon\sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q^T b$$

$\rightarrow x_1, x_2$

$L y = e_1$

$\begin{bmatrix}1 \\ l_{21} & 1 \\ l_{31} & \cdots \\ \vdots \\ \vdots & \cdots\end{bmatrix}$

$y_1 = 1$

3 $y_2 = 0 - l_{21} 1 = -l_{21} \to |y_2| = |l_{21}| \le 1$

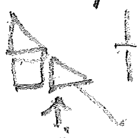
$y_i = 0 - \sum_{j=1}^{i-1} l_{ij} y_j$

suppose $|y_j| \le 1$ for $\forall j = 1, \dots, i-1$ NB this is true for $i=2$

then $|y_i| \le \sum_{j=1}^{i-1} |l_{ij}||y_j| \le \sum_{j=1}^{i-1} |l_{ij}| = 1$

ii ~~the~~ The inverse of L follows from solving $LY = I$. We solved already the first column. ~~Every other column~~
Consider the i-th column

$L y = e_i$

 $y = \begin{bmatrix}0 \\ 1 \\ \vdots \\ \vdots\end{bmatrix}$

$\uparrow i$ also diag dom next

3 so $y = \begin{matrix}0 \\ 1\end{matrix} \leftarrow$ all less than 1 in magn.

$\to |(L^{-1})_{ij}| \le 1$

iii since $LU = A$ we have $U = L^{-1}A$

$\|U\|_\infty \le \|L^{-1}\|_\infty \|A\|_\infty \le n \|A\|_\infty$

1 since $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$

$\to \|L^{-1}\|_\infty \le n$

V. $\|\delta A\|_\infty \leq \| \quad - \|$

triangle inequality

$\leq nu(3\|A\|_\infty + 5\|L\|_\infty\|U\|_\infty) + O(u^2)$

diag elem with 1 on diagonal so

$\|L\|_\infty \leq 2$

21

$\|\delta A\|_\infty \leq nu(3\|A\|_\infty + 10\|U\|_\infty) + O(u^2)$

V Using 3 we have

b) $\|\delta A\| \leq nu(\cancel{3} 3 + 10n)\|A\| + O(u^2)$

VI It holds that

$$(1)\ \frac{\|\Delta x\|}{\|x\|} \leq \frac{K(A)}{1 - K(A)\frac{\|\delta A\|}{\|A\|}}\left(\frac{\|\delta A\|_\infty}{\|A\|_\infty} + \underbrace{\frac{\|\delta b\|}{\|b\|}}_{=0}\right)$$

so from V $\frac{\|\delta A\|}{\|A\|} \leq C_n u$

2

now we find

$$(1)\ \frac{\|\Delta x\|}{\|x\|} \leq K(A)\, C_n\, u + O(u^2)$$

2 a (i) Show that the characteristic equation of the $n\times n$, n even matrix

vertrek uit huisje

$$A(\varepsilon) = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & 1 \\ \varepsilon & & & 1 \end{bmatrix}$$

is given by $(\lambda - 1)^n - \varepsilon = 0$

3

Answer

Use the last line to evaluate the determinant of $A - \lambda I$

$$-\varepsilon 1 + (1-\lambda)^n = 0 \Rightarrow (\lambda - 1)^n = \varepsilon$$

ii Consider the problem $(\lambda - 1)^n = d$.

Give the definition of the relative condition number for this problem ~~and~~ (express it in λ, n, and d)

Answer

$$K_{abs} = \max_{\Delta d \neq 0} \frac{\Delta\lambda}{\Delta d}$$

4

$$\max_{\Delta d} \frac{|\Delta d|}{|\lambda| |\Delta d|} =$$

$$\begin{cases} (\lambda + \Delta\lambda - 1)^n = d + \Delta d & \rightarrow \quad \lambda + \Delta\lambda - 1 = \sqrt[n]{d + \Delta d} \\ (\lambda - 1)^n = d & \rightarrow \quad \lambda - 1 = \sqrt[n]{d} \end{cases}$$

$$(1) \quad \Delta\lambda = \sqrt[n]{d + \Delta d} - \sqrt[n]{d} \quad \underline{d \neq 0}$$

$$= \sqrt[n]{d} \sqrt[n]{1 + \frac{\Delta d}{d}} - 1 = \sqrt[n]{d} \left[\left(e^{\Delta d / d}\right)^{1/n} - 1 \right]$$

$$= \sqrt[n]{d} \left(e^{\Delta d / nd} - 1 \right) = \sqrt[n]{d} \, \frac{\Delta d}{nd}$$

$$(1) \quad \Delta\lambda \overset{d=0}{=} \sqrt[n]{\Delta d}$$

vervolg 2b

Rel. cond. number

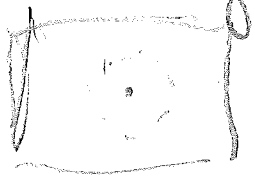
$$\max_{\Delta d} \frac{|\Delta d|}{|\lambda|\,|\Delta d|}$$

or $\frac{d}{d\Delta d}\sqrt[n]{d+\Delta d} = \frac{1}{n} d^{\left(\frac{1}{n}-1\right)}$

$$\begin{cases} \dfrac{\sqrt[n]{d}}{n\,d \cancel{/(1+\sqrt[n]{d})}} & d \neq 0 \ (2) \\[2ex] \dfrac{\sqrt[n]{\Delta d}}{\Delta d \quad 1} = \Delta d^{\frac{1}{n}-1} \end{cases}$$

K_{abs}

2a(iii) In Matlab the eigenvalues of $A(\varepsilon)$ are computed and we show the [illegible] results in the following plots. for n=10



3 Explain these results.

Answer

This can be explained from

$$(\lambda-1)^n = \varepsilon$$

Here ε is of the order of $u \approx 10^{-16}$

$$(\lambda-1)^{10} \cong 10^{-16}$$

$$\lambda-1 = 10^{-1.6} \cdot e^{i2\pi k/10}$$

This is about what we observe

2 b) The QR method ~~for~~ to find the eigenvalues of a matrix A is defined by the following iteration.

$$A^{(0)} = A$$

$$\left.\begin{aligned} Q^{(i)} R^{(i)} &= A^{(i)} \\ A^{(i+1)} &= R^{(i)} Q^{(i)} \end{aligned}\right] \quad i = 0, 1, 2 \ldots$$

i Show that $A^{(i)}$ is similar to A for all i

Answer

2 $A^{(i+1)} = Q^{(i)T} A^{(i)} Q^{(i)}$

hence $A^{(i+1)}$ similar to $A^{(i)}$ similar to $A^{(0)} = A$

ii Let $Y_{i-1} = A^{i}$ and $Q_{i-1} R_{i-1} = Y_{i-1}$

Show that $A^{(i)} = Q_{i-1}^{T} A Q_{i-1}$.

4 Answer. By induction

$Y_0 = A \qquad Q_0 R_0 = Y_0 = A$

$A^{(1)} = Q_0^T A^{(0)} Q_0 \qquad$ ok

Assume it is true for i. Show that is true for $i+1$

$$A^{(i+1)} = Q^{(i)T} A^{(i)} Q^{(i)} = Q^{(i)T} Q_{i-1}^T A Q_{i-1} Q^{(i)}$$

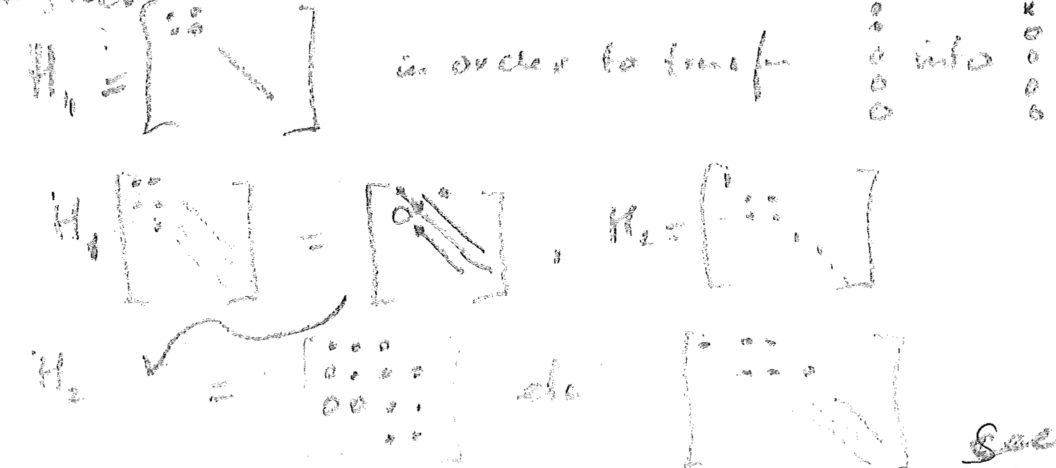
now $Q_{i-1} Q^{(i)} = Q_{i-1} A^{(i)} (R^{(i)})^{-1} = A Q_{i-1} (R^{(i)})^{-1}$

$$= A Y_{i-1} R_{i-1}^{-1} (R^{(i)})^{-1} = Y_i \hat{R}^{-1}$$

iii If A is a tridiagonal and symmetric matrix, then $A^{(i)}$ is tridiagonal for all i. Show this.

Answer

4



c Show that there exist a generalized Schur form of R

$$AZ = YS$$
$$BZ = YT$$

See next sheet

4 where both Z and Y are orthogonal and S, T upper triangular (You may assume A and B both nonsingular)

Answer

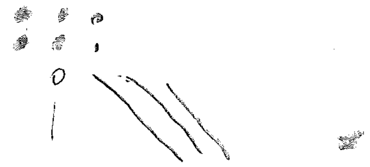
$$B^{-1}AZ = B^{-1}YS = ZT^{-1}S$$

2 So Z is the orthogonal matrix which gives the Schur form of $B^{-1}A$

So Y is the orthogonal matrix which follows from a QR fact of AZ

2 solve $Ax = \lambda Bx \iff T^{-1}Sy = \lambda y$ using QR method

$R H_1 =$



etc

However $A^{(k+1)}$ should be symmetric $\Rightarrow$ upper diag zero!

Datum:

Problem 3

~~Derive the first~~ 3

~~Legendre polynomial~~ On the interval $[0,1]$ we consider the basis for

with $\{1, x, x^2, \ldots\}$

inner prod: $(f,g) = \int_0^1 f\, g\, dx$

a Derive the first three orthogonal basis functions

$$\psi_0 = \hat{\psi}_0 = 1$$

$$\hat{\psi}_1 = x - \alpha \cdot \psi_0 \quad \text{s.t.} \quad (\hat{\psi}_1, \psi_0) = 0$$

$$\rightarrow (x, \psi_0) - \alpha(\psi_0, \psi_0) = 0$$

$$\alpha = \frac{(x, \psi_0)}{(\psi_0, \psi_0)} = \frac{(x, 1)}{(1, 1)} = \frac{\int_0^1 x\,dx}{1} = \frac{1}{2}$$

$$\psi_1 = \hat{\psi}_1 = x - \frac{1}{2}$$

$$\hat{\psi}_2 = x^2 - \alpha \psi_0 - \beta \psi_1 \quad \text{s.t.} \quad (\psi_2, \psi_0) = 0, \quad (\psi_2, \psi_1) = 0$$

4

$$\Rightarrow (x^2, \psi_0) - \alpha(\psi_0, \psi_0) = 0 \rightarrow \alpha = \frac{(x^2, \psi_0)}{(\psi_0, \psi_0)} = \frac{\int_0^1 x^2 dx}{1} = \frac{1}{3}$$

$$(x^2, \psi_1) - \beta(\psi_1, \psi_1) = 0 \rightarrow \beta = \frac{(x^2, \psi_1)}{(\psi_1, \psi_1)} = \frac{\int_0^1 x^2(x - \frac{1}{2})dx}{\int_0^1 (x-\frac{1}{2})^2 dx}$$

$$x^2 - x + \frac{1}{4}$$

$$\beta = \frac{\frac{1}{4} - \frac{1}{6}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} = \frac{\frac{1}{12}}{\frac{1}{12}} = 1$$

$$\psi_2 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

b Show that the zero's of the quadratic polynomial are $\frac{1 \pm \sqrt{\frac{1}{3}}}{2} = \frac{1}{2} \pm \frac{1}{6}\sqrt{3}$.

2

~~1a~~ c

Derive that using the first n terms of the original basis for the least squares approximation of a real function f we end up with

$$A c = b$$

where $a_{ij} = \frac{1}{i+j+1}$ and $b_i = (f, x^i)$

$$\min_{c_j} \| f - \sum_{j=0}^{n} c_j x^j \|_2^2 = \frac{1}{}$$

$$\frac{\partial}{\partial c_i} \| \quad \|^2 = 0 \qquad i = 0, \ldots, n$$

$$\frac{\partial}{\partial c_i} (\quad , \quad) = 0 \qquad " \qquad "$$

$$(x^i, \quad) + (\quad , x^i) = 0 \quad \ldots \quad "$$

may be interchanged

$$(x^i, -) = 0 \qquad \frac{2}{}$$

$$(x^i, f) - \sum_{j=0}^{n} (x^i, x^j) c_j = 0 \qquad i = 0, \ldots, n$$

$$\sum_{j=0}^{n} a_{ij} c_j = b_i$$

where $a_{ij} = (x^i, x^j) = \int_0^1 x^{i+j} \, dx = \frac{1}{i+j+1}$

$b_i = (x^i, f) = (f, x^i)$

1

~~1~~ 4

d Suppose we use the basis $\{\varphi_0, \ldots, \varphi_n\}$. Least squares min also leads to a linear system here which here. Which one

- In the above x^i is replaced by φ_i and x^j by φ_j

~~1~~ 3

We now get

$$a_{ij} = (\varphi_i, \varphi_j) = \begin{cases} (\varphi_i, \varphi_i) & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$$

$$b_i = (f, \varphi_i)$$

e The minimisations in both b and c lead ~~to~~ to a polynomial approximation of f. Why are these polynomials equal?

3 In the LS method we are minimizing the distance to a space.
Since the latter is only a orthogonal basis ~~both bases span~~ the same space, we have ~~to~~ in the end the same approximation.

f Numerically, using the orthogonal polynomials is to be favoured, why.

Because ~~B~~ in the former case the matrix gets quickly nearly singular, while the latter is not.
2 This means that the coefficients are much less determined.

g The Gauss method for the integration of

$$\frac{dy}{dt} = f(t, y)$$

is give by the Butcher tableau

$$\begin{array}{c|cc} \frac{1}{2} - \frac{1}{6}\sqrt{3} & \cdot & \cdot \\ \frac{1}{2} + \frac{1}{6}\sqrt{3} & \cdot & \cdot \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

How are the coefficients b_1, b_2 (c_1, c_2) ~~related~~ derived?

(1) + what is the order of accuracy? $\to 2n+1,\ n=2 \to 5$

compound Gauss: $2n,\ n=2 \to 4$

Answer

One writes the ODE in integral form

$$\int_{t_n}^{t_{n+1}} \frac{dy}{dt}\, dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

3 $\Rightarrow$ $$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

4 (2) The integral is approximated by the Gauss Legendre integral rule

$$\int_{t_n}^{t_{n+1}} f(t, y(t))\, dt = \int_{t_n}^{t_{n+1}} f(t_0, y(t_0))\, l_0(t) + f(t_1, y(t_1))\, l_1(t)\, dt$$

Specific to b_1, b_2

where t_0 and t_1 are the zero's of the Legendre polynomial and given by c_1 and c_2

The coefficients b_1, b_2 follow from

$$\int_{t_n}^{t_{n+1}} l_0(t)\, dt \quad \text{and} \quad \int_{t_n}^{t_{n+1}} l_1(t)\, dt$$

From the symmetry of the zeros of t_0 and t_1 wrt the middle of the interval we will have that $l_1(t_{n+1} - s) = l_0(t_n + s)$

Hence the integrals will be the same and since $b_1 + b_2 = 1 \Rightarrow b_1 = b_2 = \frac{1}{2}$

h Let $f(t,y) = y^2$, which system has to be solved in each time step

$$k_1 = \left(u_n + \tfrac{1}{4} h k_1 + \tfrac{1}{4} - \tfrac{1}{6}\sqrt{3}\, k_2\right)\left(1 + u_n + \tfrac{1}{4} \cdots\right)$$

2

$$k_2 = \left(u_n + \left(\tfrac{1}{4} + \tfrac{1}{6}\sqrt{3}\right) k_1 + \tfrac{1}{4} h k_2\right)\left(1 + u_n \cdots\right)$$

?